ENTANGLED STATE GENERATION IN A LINEAR COUPLING COUPLER

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Received on 22/5/2020, accepted for publication on 8/7/2020

Abstract: The nonlinear coupler, which consists of two nonlinear oscillators linearly coupled together and one or two of these oscillators excited by external coherent fields, is investigated. We show that evolution of the nonlinear coupler is possible closed in a finite set of *n*-photon Fock states and can create Bell-like states. Especially, the entropy of entanglement and the Bell-like states vary dramatically with the different initial conditions are discussed. These results are compared with that obtained previously in the literature.

Keywords: Kerr nonlinear coupler; Bell-like state; entropy of entanglement.

1. Introduction

Scientists are interested in two-mode nonlinear couplers, which are introduced by Jensen [1] because of their wide applicability. The nonlinear optical couplers, which rely on Kerr effect, have drawn exceptional care about both classical [1] and quantum [2] systems. The Kerr nonlinear couplers can display changes of effects as self-switching and self-trapping. For quantum fields, they are able to also create squeezed light and sub-Poissonian [3]. It is also researched on the probabilities of creating entangled states in Kerr nonlinear couplers [4]. Kerr nonlinear couplers involve two nonlinear oscillators interacting linear [5] and nonlinear [6] together. The models are advance in couplers with three nonlinear oscillators [7], the three-qubit models in phenomena of quantum steering [8], the model of three interacting qubits [9].

In this paper, we investigate Kerr nonlinear couplers including two quantum nonlinear oscillators linearly coupled together in which one or two of these oscillators excited by external classical fields and extend the consideration for all initial conditions of the motion equations of complex probability amplitudes. We show that the Bell-like states can be created in the Kerr nonlinear couplers under suitable conditions. We also compare the abilities to create Bell-like states by the nonlinear couplers pumped in one and two modes for different initial conditions of the motion equations.

2. The Kerr nonlinear coupler

2.1. The Kerr nonlinear coupler pumped in one mode

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The Kerr nonlinear coupler, which involves two nonlinear oscillators linearly interacted together, and one of these oscillators linearly interacts to an external coherent field, is studied. Therefore, the system might be depicted by the Hamiltonian [10] with the form as

$$\hat{H} = \omega_a \hat{a}^+ \hat{a} + \omega_b \hat{b}^+ \hat{b} + \frac{\chi_a}{2}\left(\hat{a}^+\right)^2 \hat{a}^2 + \frac{\chi_b}{2}\left(\hat{b}^+\right)^2 \hat{b}^2 + \varepsilon \hat{a}^+ \hat{b} + \varepsilon^* \hat{a} \hat{b}^+ + \alpha \hat{a}^+ + \alpha^* \hat{a}, \quad (1)$$

here $\hat{a}^+\left(\hat{b}^+\right)$ and $\hat{a}\left(\hat{b}\right)$ are bosonic creation and annihilation operators, corresponding to the *a* (*b*) mode of the nonlinear oscillators, respectively; χ_a (χ_b) is Kerr nonlinearity of the mode *a* (*b*); the parameters α and ε are the external coherent field for the mode *a* and the oscillator-oscillator coupling strength, respectively.

The evolution of our system without damping processes can be represented in the *n*-photon Fock basis states with the following form

$$\left|\psi(t)\right\rangle = \sum_{m,n=0}^{\infty} c_{mn}(t)\left|mn\right\rangle, \quad (2)$$

in which $c_{mn}(t)$ are the complex probability amplitudes of the system.

By using the formalism of the nonlinear quantum scissors discussed in [5], we show that the time-dependent wave function of our system can be truncated into the simple form as

$$\left|\psi(t)\right\rangle_{cut} = c_{00}^{(ij)}(t)\left|00\right\rangle + c_{01}^{(ij)}(t)\left|01\right\rangle + c_{10}^{(ij)}(t)\left|10\right\rangle + c_{11}^{(ij)}(t)\left|11\right\rangle, \quad (3)$$

$i, j = 0{,}1$ are the sign of oscillator modes, which are initially in states $\left|ij\right\rangle$.

Using the Schrödinger equation, the motion equations of the complex probability amplitudes can be depicted by the equations as

$$\begin{aligned}
i\frac{d}{dt}c_{00}^{(ij)}(t) &= \alpha^* c_{10}^{(ij)}(t),\\
i\frac{d}{dt}c_{01}^{(ij)}(t) &= \varepsilon^* c_{10}^{(ij)}(t) + \alpha^* c_{11}^{(ij)}(t),\\
i\frac{d}{dt}c_{10}^{(ij)}(t) &= \varepsilon c_{01}^{(ij)}(t) + \alpha c_{00}^{(ij)}(t),\\
i\frac{d}{dt}c_{11}^{(ij)}(t) &= \alpha c_{01}^{(ij)}(t).
\end{aligned} \quad (4)$$

By supposing that α and ε are real and for the time t = 0, both modes are originally in vacuum states ($\left|\psi(t=0)\right\rangle_{cut} = \left|00\right\rangle$), then the solutions of Eqs. (4) grow into exactly the same as those in [5]:

$$\begin{aligned}
c_{00}^{(00)}(t) &= \frac{1}{2\gamma}\left[(\gamma-\varepsilon)\cos\frac{\Delta_1 t}{2}+(\gamma+\varepsilon)\cos\frac{\Delta_2 t}{2}\right],\\
c_{01}^{(00)}(t) &= \frac{\alpha}{\gamma}\left(\cos\frac{\Delta_1 t}{2}-\cos\frac{\Delta_2 t}{2}\right),\\
c_{10}^{(00)}(t) &= -\frac{i(\gamma+\varepsilon)\Delta_2}{4\alpha\gamma}\left(\sin\frac{\Delta_1 t}{2}+\sin\frac{\Delta_2 t}{2}\right),\\
c_{11}^{(00)}(t) &= -\frac{i}{2\gamma}\left(\Delta_2\sin\frac{\Delta_1 t}{2}-\Delta_1\sin\frac{\Delta_2 t}{2}\right).
\end{aligned} \tag{5}$$

On the other hand, by assuming that for the time $t = 0$, one mode is originally in vacuum state and other mode is in single-photon Fock state ($|\psi(t=0)\rangle_{cut} = |01\rangle$), we get the solutions of Eqs. (4) for $c_{mn}^{(ij)}$, $m,n = 0,1$ in the form as

$$\begin{aligned}
c_{00}^{(01)}(t) &= c_{01}^{(00)}(t),\\
c_{01}^{(01)}(t) &= \frac{1}{2\gamma}\left[(\gamma+\varepsilon)\cos\frac{\Delta_1 t}{2}+(\gamma-\varepsilon)\cos\frac{\Delta_2 t}{2}\right],\\
c_{10}^{(01)}(t) &= \frac{i}{2\gamma}\left(\Delta_2\sin\frac{\Delta_1 t}{2}-\Delta_1\sin\frac{\Delta_2 t}{2}\right),\\
c_{11}^{(01)}(t) &= c_{10}^{(00)}(t),
\end{aligned} \tag{6}$$

where $\Delta_1 = \sqrt{2[2\alpha^2+\varepsilon^2+\varepsilon\gamma]}$, $\Delta_2 = \sqrt{2[2\alpha^2+\varepsilon^2-\varepsilon\gamma]}$, $\gamma = \sqrt{4\alpha^2+\varepsilon^2}$.

We now examine the evolution of our system for the cases when the modes are primarily in states $|10\rangle$ and $|11\rangle$. Therefore, the evolution of the system for these initial states has the form as

$$\left|\psi^{(10)}(t)\right\rangle_{cut} = c_{10}^{(00)}|00\rangle + c_{10}^{(01)}|01\rangle + c_{01}^{(01)}|10\rangle + c_{01}^{(00)}|11\rangle, \tag{7}$$

and

$$\left|\psi^{(11)}(t)\right\rangle_{cut} = c_{11}^{(00)}|00\rangle + c_{10}^{(00)}|01\rangle + c_{01}^{(00)}|10\rangle + c_{00}^{(00)}|11\rangle, \tag{8}$$

and the entropies of entanglement are also easily obtained as

$$E^{(11)}(t) = E^{(00)}(t), \qquad E^{(10)}(t) = E^{(01)}(t). \tag{9}$$

2.2. The Kerr nonlinear coupler pumped in two modes

The Kerr nonlinear coupler pumped in two modes is similar to the coupler pumped in single mode, except both modes of this coupler are coupled by external coherent fields. Hence, the Hamiltonian depicting such system has the following form

$$\hat{H} = \frac{\chi_a}{2}(\hat{a}^+)^2\hat{a}^2 + \frac{\chi_b}{2}(\hat{b}^+)^2\hat{b}^2 + \varepsilon\hat{a}^+\hat{b} + \varepsilon^*\hat{a}\hat{b}^+ + \alpha\hat{a}^+ + \alpha^*\hat{a} + \beta\hat{b}^+ + \beta^*\hat{b}. \tag{10}$$

This Hamiltonian is similar to the one defined by (1), except for the term $\beta\hat{b}^{+}+\beta^{*}\hat{b}$, in which β is the coupling strength of the mode *b* with an external coherent field.

In this case, we also use the Schrödinger equation and obtain the motion equations of the complex probability amplitudes in the form

$$\begin{aligned}
i\frac{d}{dt}c_{00}^{(ij)}(t) &= \alpha^{*}c_{10}^{(ij)}(t)+\beta^{*}c_{01}^{(ij)}(t),\\
i\frac{d}{dt}c_{01}^{(ij)}(t) &= \varepsilon^{*}c_{10}^{(ij)}(t)+\alpha^{*}c_{11}^{(ij)}(t)+\beta c_{00}^{(ij)}(t),\\
i\frac{d}{dt}c_{10}^{(ij)}(t) &= \varepsilon c_{01}^{(ij)}(t)+\alpha c_{00}^{(ij)}(t)+\beta^{*}c_{11}^{(ij)}(t),\\
i\frac{d}{dt}c_{11}^{(ij)}(t) &= \alpha c_{01}^{(ij)}(t)+\beta c_{10}^{(ij)}(t).
\end{aligned} \tag{11}$$

By solving these equations for all initial states of the modes, we shall obtain their solutions similar to those for coupler pumped in single mode. Because of the limitation of the volume in this work scale, we focus only on studying the generation of Bell-like states in the next section, whereas their mathematical details will not be presented.

3. The generation of Bell-like states in the Kerr nonlinear coupler

The entropy of entanglement of our system is defined as in [5]:

$$E^{(ij)}(t)=-\lambda.\log_2\lambda-(1-\lambda).\log_2(1-\lambda), \tag{12}$$

where $\lambda=\frac{1+\sqrt{1-\left(C^{(ij)}\right)^2}}{2}$ and $C^{(ij)}=2\left|c_{00}^{(ij)}(t)c_{11}^{(ij)}(t)-c_{01}^{(ij)}(t)c_{10}^{(ij)}(t)\right|$.

The truncation state in (3) can be represented in the Bell basis states in the form:

$$\left|\psi(t)\right\rangle_{cut}=\sum_{l=1}^{4}b_l^{(ij)}(t)\left|B_l^{(ij)}\right\rangle, \tag{13}$$

where Bell states are expanded by the Bell-like states with the form as

$$\begin{aligned}
\left|B_1^{(ij)}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|00\right\rangle+i\left|11\right\rangle\right)^{(ij)}, &\quad \left|B_2^{(ij)}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|11\right\rangle+i\left|00\right\rangle\right)^{(ij)},\\
\left|B_3^{(ij)}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|01\right\rangle-i\left|10\right\rangle\right)^{(ij)}, &\quad \left|B_4^{(ij)}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|10\right\rangle-i\left|01\right\rangle\right)^{(ij)}.
\end{aligned} \tag{14}$$

By using (3) and (13), the coefficients $b_l^{(ij)}$ can be achieved in the following form

$$\begin{aligned}
b_1^{(ij)} &= \frac{1}{\sqrt{2}}\left(c_{00}^{(ij)}(t)-ic_{11}^{(ij)}(t)\right), &\quad b_2^{(ij)} &= \frac{1}{\sqrt{2}}\left(c_{11}^{(ij)}(t)-ic_{00}^{(ij)}(t)\right),\\
b_3^{(ij)} &= \frac{1}{\sqrt{2}}\left(c_{01}^{(ij)}(t)+ic_{10}^{(ij)}(t)\right), &\quad b_4^{(ij)} &= \frac{1}{\sqrt{2}}\left(c_{10}^{(ij)}(t)+ic_{01}^{(ij)}(t)\right).
\end{aligned} \tag{15}$$

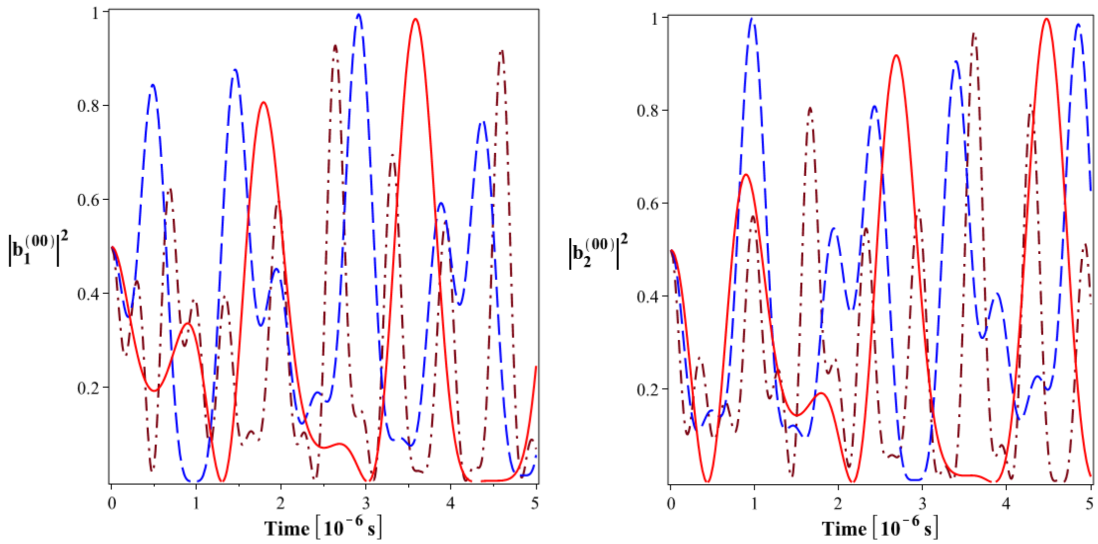

Figure 1: *The probabilities to the system exist in the Bell-like states* $|B_1^{(00)}\rangle$ *and* $|B_2^{(00)}\rangle$ *for the coupler pumped in one mode with* $\alpha=\varepsilon=10^6\pi$ *rad/s,* $\beta=0$ *(solid line) and in two modes with* $\alpha=\varepsilon=\beta=10^6\pi$ *rad/s (dashed line) and* $\alpha=\varepsilon=\beta/2=10^6\pi$ *rad/s (dashed dotted line).*

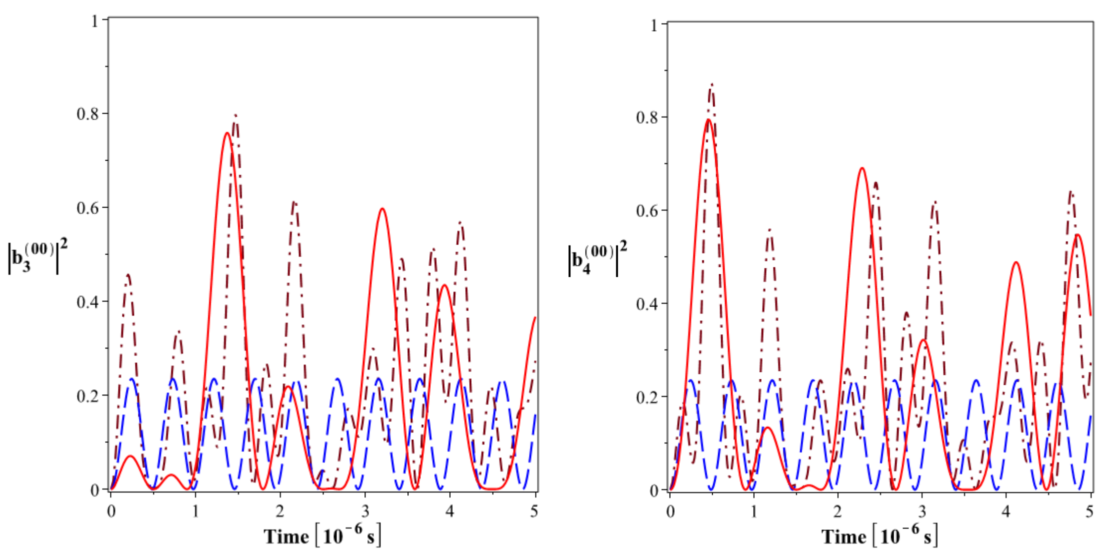

Figure 2: *The probabilities to the system exist in the Bell-like states* $|B_3^{(00)}\rangle$ *and* $|B_4^{(00)}\rangle$ *for the coupler pumped in one mode with* $\alpha=\varepsilon=10^6\pi$ *rad/s,* $\beta=0$ *(solid line) and in two modes with* $\alpha=\varepsilon=\beta=10^6\pi$ *rad/s (dashed line) and* $\alpha=\varepsilon=\beta/2=10^6\pi$ *rad/s (dashed dotted line).*

Here, the figures of probabilities, which maintain the system in the Bell-like states $|B_1^{(01)}\rangle$ and $|B_2^{(01)}\rangle$ is not presented, as we have already obtained $|b_1^{(01)}|^2=|b_4^{(00)}|^2$ and

$\left|b_2^{(01)}\right|^2 = \left|b_3^{(00)}\right|^2$. The probabilities to the system exist in the Bell-like states in which the modes are originally in states $|00\rangle$ and $|01\rangle$ are presented in figures from 1 to 3. When the coupler pumped in one mode ($\beta = 0$), for the modes are originally in states $|00\rangle$, we get the same results as the ones in [5] (Figs. 1 and 2). For the modes are primarily in states $|01\rangle$, the probabilities for the creation of the maximally entangled states as well as a function of time for the single-mode control couplers and the system can be also generated Bell-like states for the states $|B_3^{(01)}\rangle$ and $|B_4^{(01)}\rangle$ (Fig. 3). When the coupler pumped in two modes, the system can be generated the maximally entangled states for the states $|B_1^{(00)}\rangle$, $|B_2^{(00)}\rangle$ (Fig. 1) and $|B_3^{(01)}\rangle$, $|B_4^{(01)}\rangle$ (Fig. 3), but it cannot be created the maximally entangled states for the states $|B_3^{(00)}\rangle$ and $|B_4^{(00)}\rangle$ (Fig. 2). Especially, when $\beta = \alpha$, the maximum values of the probabilities are the greatest for states $|B_1^{(00)}\rangle$, $|B_2^{(00)}\rangle$ and $|B_4^{(01)}\rangle$, whereas they are the smallest for states $|B_3^{(00)}\rangle$ and $|B_4^{(00)}\rangle$. Moreover, when the parameter $\beta > \alpha$, the probabilities for the existence of the system in states $|B_1^{(00)}\rangle$, $|B_2^{(00)}\rangle$ and $|B_3^{(01)}\rangle$, $|B_4^{(01)}\rangle$ decrease, while the probabilities $|B_3^{(00)}\rangle$ and $|B_4^{(00)}\rangle$ increase.

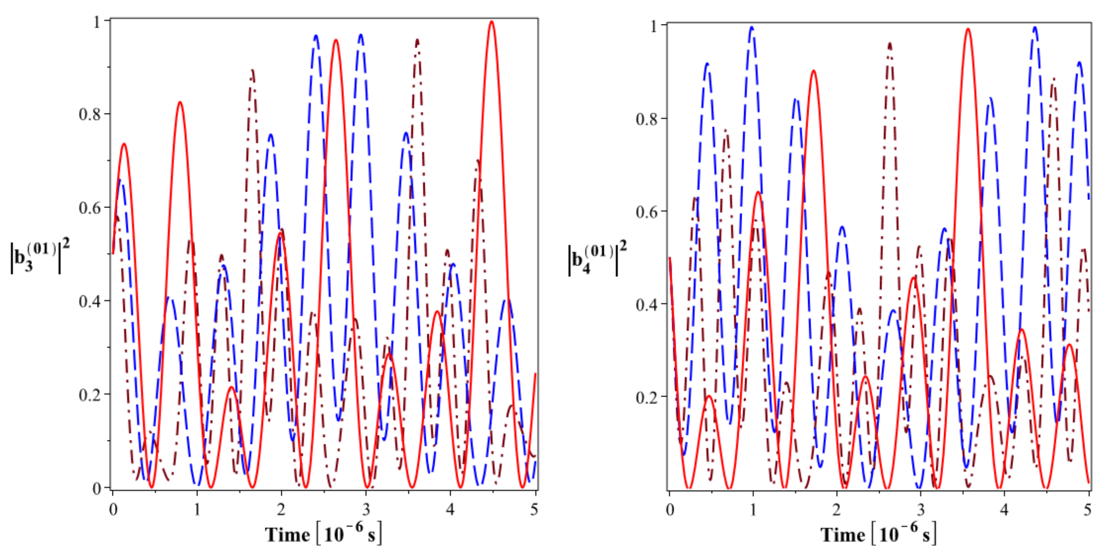

Figure 3: *The probabilities to the system exist in the Bell-like states* $|B_3^{(01)}\rangle$ *and* $|B_4^{(01)}\rangle$ *for the coupler pumped in one mode with* $\alpha = \varepsilon = 10^6\pi$ *rad/s,* $\beta = 0$ *(solid line) and in two modes with* $\alpha = \varepsilon = \beta = 10^6\pi$ *rad/s (dashed line) and* $\alpha = \varepsilon = \beta/2 = 10^6\pi$ *rad/s (dashed dotted line).*

The entropies of entanglement of the system are shown in figure 4. The results of $E^{(00)}$ for the coupler pumped in single mode ($\beta = 0$) and in two modes ($\beta = \alpha$) are the same as those in [5]. The entangled entropies $E^{(00)}$ and $E^{(01)}$ are progressing in cycles of time and they approximately are equal to 1 ebit for maximally entangled states, whereas they are equal to zero for separable states. For $\beta = \alpha$, the maximum values of the $E^{(00)}$

and $E^{(01)}$ are the highest while they are the lowest for $\beta > \alpha$. Furthermore, the entropy of entanglement $E^{(01)}$ has more maxima than $E^{(00)}$, which means that $E^{(01)}$ oscillates faster than $E^{(00)}$. Consequently, the maximally entangled states and the entropy of entanglement vary considerably for the modes, which are initially in different states.

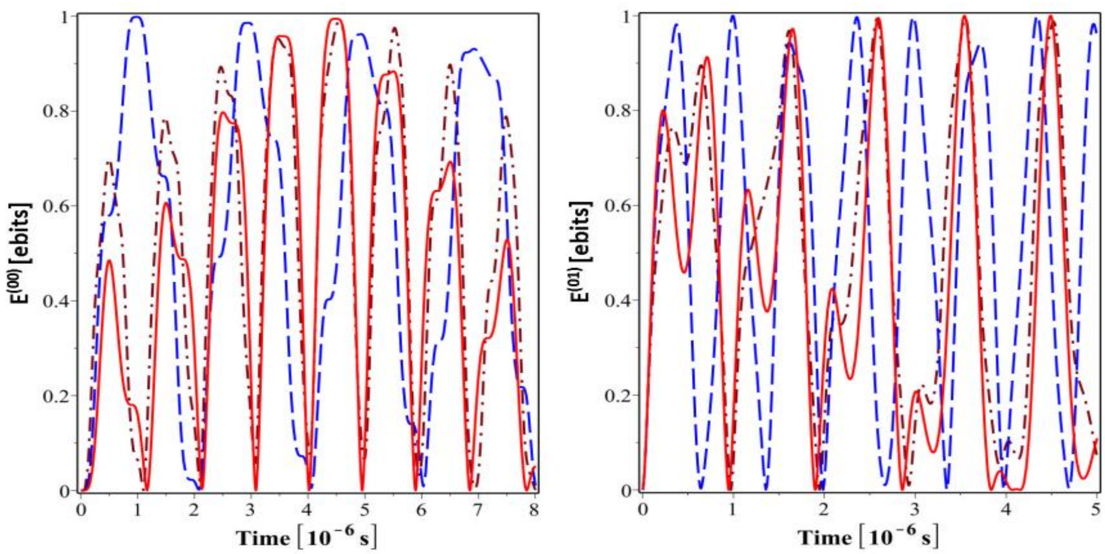

Figure 4: *Evolution of the entropies of entanglement* $E^{(00)}$ *and* $E^{(01)}$ *for the coupler pumped in one mode with* $\alpha = \varepsilon = 10^6\pi$ *rad/s,* $\beta = 0$ *(solid line) and in two modes with* $\alpha = \varepsilon = \beta = 10^6\pi$ *rad/s (dashed line) and* $\alpha = \varepsilon = \beta/2 = 10^6\pi$ *rad/s (dashed dotted line).*

For brevity, we do not present the figures of the probabilities for the system to exist in Bell-like states, and the entropies of entanglement for the modes in states $|10\rangle$ and $|11\rangle$ because they are shown in figures from 1 to 4 for the modes are initially in states $|00\rangle$ and $|01\rangle$.

4. Conclusion

In this work, we have investigated the model of the Kerr nonlinear coupler consisting of two nonlinear oscillators linearly coupled at one another and one or two of these oscillators are linear interaction with external classical fields. By using the method of nonlinear quantum scissors, we have achieved the probabilities for the existence of the system in the maximally entangled states and the entropies of entanglement for the original modes in four states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. We have also shown that the Kerr nonlinear coupler creates the Bell-like states for the primary modes in all these states. Furthermore, the entangled entropy and the Bell-like states potentially vary for the modes in different states.

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TÓM TẮT

SỰ SINH TRẠNG THÁI ĐAN RỐI TRONG BỘ NỐI LIÊN KẾT TUYẾN TÍNH

Bộ nối phi tuyến gồm hai dao động tử phi tuyến liên kết tuyến tính với nhau và một hoặc hai dao động tử này được kích thích bởi các trường kết hợp ngoài được nghiên cứu một cách chi tiết. Chúng tôi chỉ ra rằng sự tiến triển của bộ nối phi tuyến này có thể được đóng trong một tập hợp hữu hạn các trạng thái Fock n-photon và có thể tạo ra các trạng thái kiểu Bell. Đặc biệt, entropy đan rối và các trạng thái kiểu Bell thay đổi một cách đáng kể với các điều kiện đầu khác nhau sẽ được thảo luận. Các kết quả này sẽ được so sánh với những kết quả tìm được trong các công trình trước đó.

Từ khóa: Bộ nối phi tuyến Kerr; trạng thái kiểu Bell; entropy đan rối.